IX. *The Construction and Properties of a new* Quadratrix *to the* Hyperbola, *By Mr* . . . Perks. *Communicated by Mr* Abr. de Moivre, *F. R. S.*

THe *Circle*, *Ellipsis* and *Hyberbola* being not Geometrically Quadrable (as infinite others) there have been two ways made use of to find their *Area*'s. 1. By *Converging Series*, whereby Approaches are made nearer and nearer, according to the exactness desir'd. 2. By *Quadratices*, that is, Mechanical Curves, which determine the Length of certain Lines, whose Squares or Rectangles give the *Area* of the Figure desir'd. Of this sort is the old *Quadratrix of Dinostratus*, by which the Circle and Ellipse are squared ; and another sort (for the same purpose) I inserted in the *Transactions* about 5 years ago. Since that, having found the Construction of a Curve, from whence (besides its own *Quadrature* and *Rectification*) the *Quadrature* of the *Hyperbola* is deriv'd, I thought the following Account might not (to some) be unacceptable.

Let A B, C D, be two straight Rulars joyned at B, and there making a right Angle. (Their length according to the largeness of the Figure you will describe.) E F is another Rular somewhat longer than A B. Near the one end E, let a little *Truckle-wheel* (represented edge-wise by *g h*, and made of a thin Plate of Brass or Iron) be fastned to the Rular by a Pin (*i*,) thorow its Center, so that the Wheel may turn about upon the Pin (*i*) tight to the Rular without joggling.

On the under side of this Rular (the side from the Eye in the Scheme) let there be pinn'd or glewed a little piece of Wood (in the form of a Quadrant, the part which is seen being mark'd *k l*) whose edge (or limb) *k l*, is an arch of a Circle of Center (*i*,) and *Radius i b* (the same with the little Wheel.) The design of this piece of Wood is, that in the several Positions of the Rular E F, the circular Limb *k l* always touching and sliding by the edge of the Rular A B, the Center of the Wheel may be always in a line (*i m*) parallel to the Rular A B.

In the Rular C D make M B = *i b* or *i k*, and at M fasten a little Pin, and another to the Rular E F near the Wheel, as at *p*. To these two Pins let be fastned the two ends of a String M R, so that its whole length (from Pin to Pin) + *p i*, be equal to the intended Axis of the Curve T W.

The Instrument being thus prepar'd, let a strong Rular S O, be fastned (or held fast) upon the Paper or Plain that the Curve is to be drawn upon. Lay the Rular E F from M towards A, and parallel to A B, so that the String lye all straight along the edge of the Rular E F from M to *p*, the point *Sk* of the Quadrantal piece of Wood resting upon the edge of the Rular A B. Then with a small Pin at M keeping the String close to the edge of the Rular E F, and with your other hand upon the end E, keeping the Wheel tight to the Paper or Plain, move the Pin String and Rular E F from M towards O, the Rular C D sliding along by the fastned Rular S O in a right line, the Wheel *g h* will by its motion describe the desired Curve T V.

Note,

Note, The Semi-diameter of the little Wheel muſt be about the *Sum* of the thickneſſes of the two Rulers E F and A B, that it may touch the Paper. Alſo it will be convenient that its edge be thin, and a little rough, that it may not ſlide flat-ways, and that it may leave a viſible impreſſion.

From this Conſtruction the following Properties are demonſtrable.

I. It is evident from the Conſtruction, that the *Sum of the Tangent and Subtangent* is every where equal to the ſame given Line (= M R + R i = T W.) for the String (firſt ſtraight at T W, afterwards making an Angle at R) being every where the ſame, the Line R i (or R p + p i) is always the Tangent, and the Remainder R M the Subtangent ; the Contact of the Wheel with the Plain, being the point of the Curve to which they belong.

II. It hence follows, that any aſſignable part of the Curve is *Rectifiable*, or equal to any aſſignable ſtraight Line. In Fig. 2. Let F A E be a part of the Curve, its Vertex F. H D *d* is the Line deſcribed by the motion of the Pin R (in Fig. 1.) and may be ſhewn to be Aſſymptote to the Curve. F H a perpendicular to H D. Let A be given point in the Curve, A D the Tangent, and B D the Subtangent to the ſame point A. Let *a* be another point in the Curve infinitely near to A. to which let *a d* be the Tangent, and *b d* the Subtangent. Draw A G, *a g* perpendicular to F H and A B, *a b* perpendicular to H D. By the Conſtruction A D + D B = *a d* + *d b*. Let a δ be made equal to *a* D, and draw D δ. Then becauſe *a d* + *b d* = A D + D B. Subtract b D and a D (or a δ) from both Sums (Equals from Equals) there remains δ d + d D = A a + B b (or C a) A a C, D d δ are like Triangles (or differing infinitely little from such) therefore C a (B b): A a :: δ d : D d. and compounding B b + A a : A a :: δ d + D d : D d. Alternating B b + A a : δ d + D d :: A a : D d. But B b + A a = δ d + D d (as is shewn above) therefore A a = D d. A a is the fluxional Particle of the Curve F A, and D d is the fluxional Particle of the Line H D: These Fluxions or Augments, being equal, and their flowing quantities beginning together, are themselves therefore equal, *viz.* F A = H D.

Let F G = x. G A (= H B) = y. A D = t. B D = S. So is the Curve F A = H D = y + S: that is, *the Curve from the Vertex to any given point therein, is equal to the Sum of its Ordinate, and Subtangent to the same point which is its second Property.*

III. The next Property (and whereupon I call it the *Hyperbolic Quadratrix*) is this, In Fig. 2. let F A E be a part of the Curve, (*&c.* as before.) F I K H is a Square upon the line F H. A I L is an Equilater Hyperbola whose Vertex is I, its Asymptotes H O, H R. its Ax H I μ. From a given point L in the Hyperbola (below its Vertex I) draw L A parallel to the Asymptote R H, intersecting the Diagonal I H in M, F H in G, and touching the Quadratrix in A. I say, that the Hyperbolic Area I L M is equal to a Rectangle, whose sides are the *Ordinate* G A, and twice F H, the Ax to the Quadratrix, that is, Trilin. I L M = 2 F H × G A.

Let F H = a, F G = x, G A = y. because of the Hyperbola G L X G H (L S) = F a q. therefore G L $= \frac{FHq}{GH}$; and L M $= \frac{FHq}{GH}$ — G H (M G) that is,

L M $= \frac{aa}{a-x} - a + x = \frac{2ax - xx}{a \text{ [illegible]}}$, and consequently

the fluxion of the Area I L M = [illegible]

In

In the Rectangle triangle A D B, A B = a − x, B D = S, A D = t = a − S; then is A D q = A B q + B D q: or a a 2 a S + SS = a a − 2 a x + x x + S S, which being reduced, gives $S = \frac{2ax - xx}{2a}$

Let *l a* be a right line suppofed infinitely near and parallel to L A, and interfecting A B in C. Becaufe of like triangles A C a, A B D; A B : B D : : A C : C a that is $a - x : S \left(= \frac{2ax - xx}{2a} \right) :: \dot{x} : \dot{y}$. therefore $\dot{y} = \frac{2ax - xx}{2aa - 2ax}\dot{x}$. Multiply each by 2 a, and 'tis $2a\dot{y} = \frac{2ax - xx}{a - x}\dot{x}$. The *Flowing quantity* of $2a\dot{y}$ is $2ay$ and the *flowing quantity* of $\frac{2ax - xx}{a - x}\dot{x}$ is the Hyperbolic Area I L M (as is fhewn before.) Thefe two Area's beginning together at F and I, and having every where equal *Fluxions*, or Augments, are therefore themfelves every where equal.

N. The Quadrature of the Trilinear Figure I L M being thus found, any other Area bounded with the Curve line I L. and any other Right Lines is alfo given.

IV. Suppofing the fame things as in the precedent Propofition, I fay, that the Area of the Quadratrix F a b H F is equal to half the fquare of F g, wanting the Cube of F g divided by fix F H, or $F\,a\,b\,H\,F = \frac{xx}{x} - \frac{xxx}{6a}$. The Fluxion of this Area is the Rectangle $C\,a\,b\,B = \overline{a - x} \times \dot{y} = \overline{a - x} \times \frac{2ax - xx}{2aa - ax}\dot{x} = x\dot{x} - \frac{xx}{2a}\dot{x}$. The flowing quantity of $x\dot{x}$ is $\frac{1}{2}xx$: And the flowing quantity

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tity of $-\frac{xx}{2a}\dot{x}$ is $-\frac{xxx}{6a}$ [as is easily shewn by bringing back these flowing quantities to their respective Fluxions.] And hence also it follows, that the whole Area continued on infinitely towards F, is *one third of the Square* F I K H; or $\frac{1}{3}$ a a. Eor supposing x = a the Area above becomes $\frac{aa}{2} - \frac{aa}{6} - \frac{aa}{3}$.

While I was considering the other Properties of this Curve, and had given some account of them to my Ingenious Friend Mr *John Colson*, he returned me a Letter with the Addition of the Quadrature of the Curves Area, which I had not then enquired into.

V. Supposing still the same things, I say that the Solid made by the conversion of the Area F a b H F about the Line H b as an Axis, is equal to a Cylinder whose Radius is F H = a, and height equal to $\frac{xx}{2a} - \frac{x^3}{2aa} + \frac{x^4}{8a^3}$. And the whole Solid made by conversiou of the whole Figure infinitely continued, is equal to an eighth part of a Cylinder, whose Radius and Height are each equal to F H or *a*.

Let $\frac{P}{D}$ express the Proportion of the Periforie and Diameter of a Circle. Then is $\frac{P}{D}$ a b quad. the Area of a Circle whose Radius Is a b. And because C a = $\dot{y}$ = $\frac{x - \frac{xx}{2a}\dot{x}}{a - }$ the fluxion of the Solid is $\frac{P}{D}$ *a b*. q. $x - \frac{xx}{2a}\dot{x}$

or

or $\frac{P}{D}\overline{a-x}^2 \times \frac{\dot{x} - \frac{xx}{2a}\dot{x}}{a-x} = \frac{P}{D}\overline{ax - \frac{3}{2}xx + \frac{x^3}{2a}}\,\dot{x}$

whose flowing quantity is $\frac{P}{D}\,\frac{axx - xxx}{2} + \frac{x^4}{8a}$. Which Solid being divided by $\frac{P}{D}aa$ (the Area of a Circle whose Radius is a) gives $\frac{xx}{2a} - \frac{xxx}{2aa} + \frac{x^4}{8aa}$ for the height of a Cylinder on the said circular Base, and equal to the Solid made by conversion of the Area F a b H F about the Line H b as an Axis. When $x = a$ (that is when the whole Figure is turn'd about its Asymptote) the height $\frac{xx}{2a} - \frac{x^3}{2aa} + \frac{x^4}{8aa}$ become $\frac{1}{8}a$

VI. The Curve surface of the Solid generated by the Conversion of the Figure F a b H F about H B, is equal to the Curve surface of a Cylinder, whose Radius is a, and height equal to $\frac{x}{2} - \frac{xx}{4a} + \frac{xxx}{12aa}$. And the whole Curve Surface of the Solid infinitely continued, is equal to *one third part of the Curve Surface of a Cylinder whose Radius and Height are equal to* F H or a. Which may be demonstrated after the manner of the precedent Proposition.

VII. The Radius of the Curvature of any Particle of the Quadratrix is $\frac{tt}{a-x}$ and this found Geometrically. In Fig. 3. F A E is the *Quadratrix*, H D the Asymptote, A D the Tangent, B D the Subtangent to a given point A. Make B V = A D. Upon V rise the perpendicular V W. from A draw A W perpendicular to the Tangent A D, till it it meet A W in W. So is A W the Radius of the Curvature at A.

VIII. This Curve may be continued on infinitely above the point F (but by a different and more operose way of Construction) whose Properties will be these. 1. The *Difference* of its Tangent and Subtangent (taking the Subtangent in the Line H S) will be always equal to the same given Line F H or *a*. That is, as t + s = a, below F, so t — s = a above F. 2. As below F the Curve Line is equal to the *Sum* of its Ordinate and Subtangent, so above, it is equal to their *Difference* or — S—y. 3. As below F, 2 a y = I L M, so above 2 a y = I λ μ. All which (and its other Properties) may be demonstrated as the Precedent *mutatis mutandis*.

IX. With a little variation in the precedent Construction may the *Logarithmick Curve* be constructed, which is also a *Quadratrix* to the Hyperbola. In Fig. 1. omitting the String M R P, let the distance M R be equal to the *Subtangent* of the intended Logarithmick Curve (which, as 'tis known, is invariable.) Stick a Pin at R in the Rular C D, to which apply the Rular E F, so that the edge of the little Quadrant *k l*, resting upon the Rular A B, the distance M i be equal to M R. Then keeping the Rular E F tight to the Pin R and Rular A B, slide the Rular C D along in a straight Line (by the Rular or Line S O.) So will the Wheel *g h* describe a part of the Logarithmick Curve T V, whose *Subtangent* is every where M R.

X. Fig. 2. Let F A E represent the *Logarithmick Curve*, whose Subtangent is equal to F H. L I λ is an Equilater Hyperbola (*&c.* as before § III.) Let F G = x, G a = y. F H (= B D) = a. G H (= L S) = a — x. A C = x C a = y. Then A C : C a : : A B : B D. that is x : y : : a

— x

$x : a :: a : \frac{aa}{a-x}$, therefore $a\dot{y} = \frac{aa}{a-x}\dot{x}$. The *Flowing quantity* of $a\dot{y}$ is ay; and the *Flowing quantity* of $\frac{aa}{a-x}\dot{x}$ is the Hyperbolick Area F I L G (for by the nature of the Hyperbola $GL = \frac{aa}{a-x}$) therefore is the Hyperbolick Area F I L G equal to a y, a Rectangle whose sides are the Subtangent (B D = F H) and Ordinate G A (as here accounted) of the Logarithmick Curve.

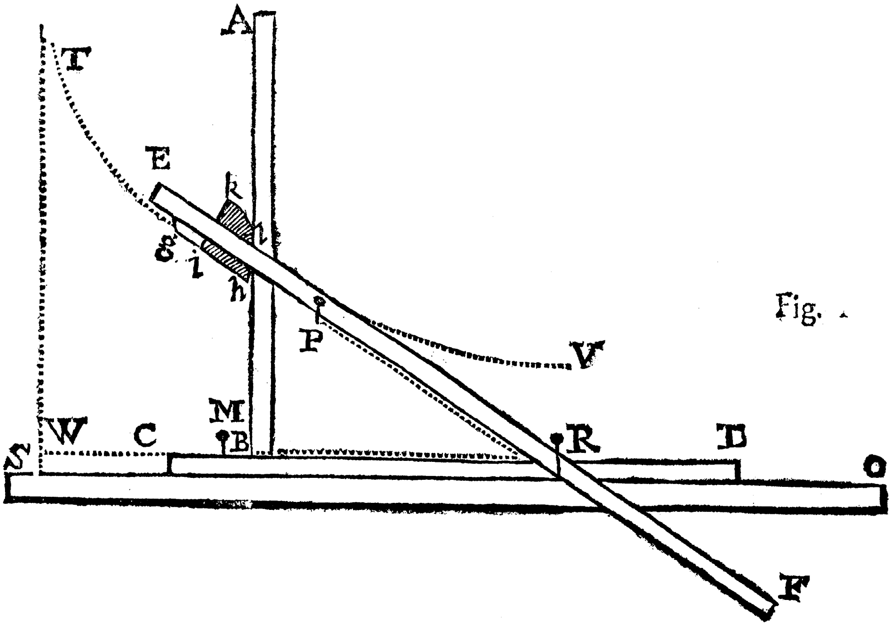

Fig. 1.

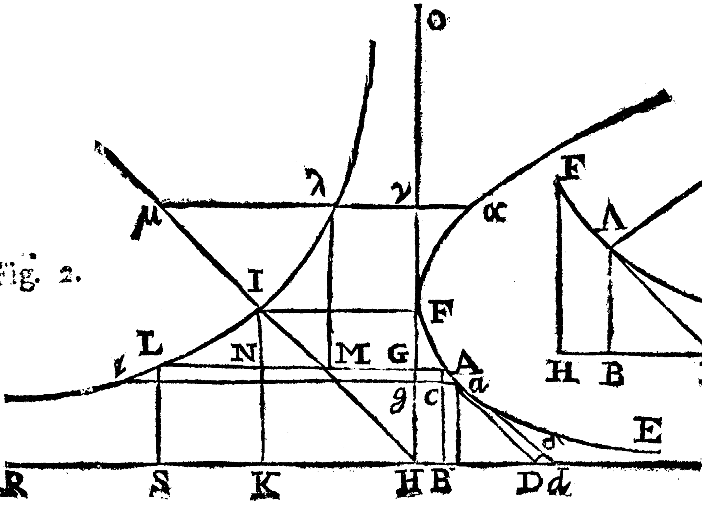

Fig. 2.

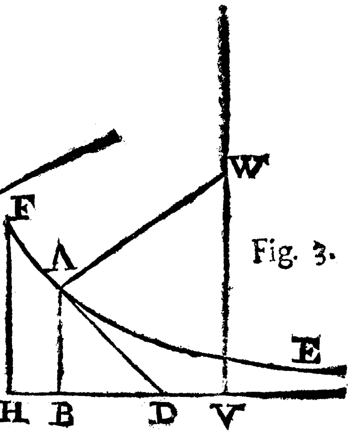

Fig. 3.

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